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Supergravity spectrum on $AdS_2 \times S^2$

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Abstract: The Kaluza-Klein spectrum of $\mathcal{N} = 2$, $D = 4$ supergravity compactified on $AdS_2 \times S^2$ is found and shown to consist of two infinite towers of $\mathrm{SU}(1, 1|2)$ representations. In addition to 'pure gauge' modes living on the boundary of AdS which are familiar from higher dimensional cases, in two dimensions there are modes (e.g. massive gravitons) which enjoy no gauge symmetry yet nevertheless have no on-shell degrees of freedom in the bulk. We discuss these two-dimensional subtleties in detail.

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1. Introduction

Following the conjecture of Maldacena [1, 2, 3] relating supergravity (or string theory) on AdS_d to a $(d-1)$-dimensional conformal field theory (see [4] for a review) there has been a revival of interest in supergravity compactifications on anti-de Sitter spaces.

The Kaluza-Klein spectrum of these compactifications should be identical to the spectrum of chiral primary operators in the dual conformal field theory. Recent papers have considered supergravity compactifications on $AdS_3 \times S^3$ [5, 6], $AdS_3 \times S^2$ [7, 8], and $AdS_2 \times S^3$ [9]. We complete the square by considering $\mathcal{N} = 2$, $D = 4$ supergravity on $AdS_2 \times S^2$. Interest in $AdS_2 \times S^2$ stems from the fact that it is the near-horizon geometry of the extreme Reissner-Nordström black hole in four dimensions [10], and because a precise elucidation of the nature of the dual $0 + 1$ dimensional conformal field theory (i.e. quantum mechanics) living on the boundary of AdS_2 is lacking. For relevant work in this direction, see [11, 10, 12].

Compactification of supergravity to two dimensions presents difficulties not present in higher-dimensional cases. One is that the equation of motion for the two-dimensional graviton $h_{\mu\nu}$ cannot be diagonalized because it involves some scalar fields which cannot be eliminated. However, far from being a problem, this is to be expected in two dimensions — conformal invariance in two dimensions prevents the definition of the Einstein frame. This presents no obstacle to determining the mass spectrum, since although these scalar fields act as sources for the two-dimensional tensor field $h_{\mu\nu}$ they do not affect its mass.

Under dimensional reduction on AdS times spheres one obtains infinite towers of Kaluza-Klein modes with on-shell degrees of freedom in the bulk, and in addition one typically finds a small number of 'pure gauge' modes which live only on the boundary of AdS. These include the doubleton multiplet of $\mathrm{SU}(2,2|4)$ in the case of AdS_5 [13] and the singleton multiplet of $\mathrm{OSp}(4|8)$ in the case of AdS_4 [14]. These modes arise from fields which can be gauged away everywhere in the bulk using residual gauge transformations (we discuss these in detail in section 3.1). In compactification to two dimensions one encounters a third type of field: those which have no on-shell degrees of freedom in the bulk not because of any gauge invariance but simply because their equations of motion impose sufficiently many constraints to eliminate the field completely. In two dimensions, massive gravitons and gravitini fall into this category. A massive graviton, for example, enjoys no gauge invariance and has $\frac{1}{2}(d+1)(d-2)$ degrees of freedom on-shell in d dimensions. By a careful analysis of the equations of motion for these massive gravitons in AdS_2 we will show that they do not give rise to independent degrees of freedom.

In section 2 we discuss the $AdS_2 \times S^2$ solution of $\mathcal{N} = 2$, $D = 4$ pure supergravity. In section 3 we determine the spectrum of bosonic Kaluza-Klein fluctuations around the $AdS_2 \times S^2$ background. Section 3.1 contains a detailed discussion of the residual gauge transformations in $AdS_2 \times S^2$. The results of section 3 are fairly simple but great care is taken to address in detail the subtleties mentioned above. Section 4 addresses the fermionic Kaluza-Klein fluctuations. Finally, in section 5 we arrange the Kaluza-Klein modes into representations of the supergroup $\mathrm{SU}(1,1|2)$.

2. The $AdS_2 \times S^2$ supergravity solution

2.1 Lagrangean

The gravity multiplet of $\mathcal{N} = 2$, $D = 4$ supergravity contains the spin-2 graviton g_{MN}, a complex spin-3/2 gravitino Ψ_M, and a spin-1 graviphoton A_M. Our notational conventions are summarized in appendix A. The lagrangean is [15]

$$\mathcal{L} = \sqrt{-g}\left[R - \frac{1}{4}F^2 - \frac{i}{2}\overline{\Psi}_M \Gamma^{MNP} \nabla_N \Psi_P - \frac{i}{4}\overline{\Psi}_M \left(F^{MN} - i\Gamma^5 * F^{MN}\right)\Psi_N\right] + \mathcal{L}_4 \,, \tag{2.1}$$

where $\mathcal{L}_4$ contains terms quartic in Ψ_M which will not be relevant for our analysis since we are only interested in fluctuations around a classical background with $\Psi_M = 0$.[1] The action is invariant under $\mathcal{N} = 2$ supersymmetry transformations, whose action on the gravitino field is

$$\delta_\epsilon \Psi_M = 2\left[\nabla_M + \frac{i}{8} F_{NP}\Gamma^{NP}\Gamma_M + \cdots\right]\epsilon \,, \tag{2.2}$$

where ϵ is a complex spinor parameter and the dots represent terms bilinear in Ψ_M which we will not need. We will make extensive use of the operator

$$D_M \equiv \nabla_M + \frac{i}{8} F_{NP}\Gamma^{NP}\Gamma_M \,. \tag{2.3}$$

2.2 Freund-Rubin ansatz

The bosonic equations of motion arising from (2.1) in a background with $\Psi_M = 0$ are the familiar Einstein-Maxwell equations

$$\begin{aligned} R_{MN} - \frac{1}{2}R g_{MN} &= \frac{1}{2}\left(F_{MP}F_N{}^P - \frac{1}{4}g_{MN}F^2\right), \\ \nabla_M F^{MN} &= 0 \,. \end{aligned} \tag{2.4}$$

It follows from the trace of Einstein's equation that $R = 0$. We take the Freund-Rubin ansatz [16]

$$F_{\mu\nu} = 0 = F_{\mu m} \,, \qquad F_{mn} = 2\epsilon_{mn} \,. \tag{2.5}$$

Assuming a product space, the Einstein equations then give

$$R_{\mu m} = 0 \,, \qquad R_{\mu\nu} = -g_{\mu\nu} \,, \qquad R_{mn} = g_{mn} \,, \tag{2.6}$$

which is consistent with the ansatz, and gives the desired $AdS_2 \times S^2$ topology, with unit radii.

[1] In particular, we do not explore the consistency of a truncation to the massless sector of the two-dimensional theory on AdS_2.

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2.3 Supersymmetry

The $AdS_2 \times S^2$ solution preserves the maximal amount of supersymmetry. The supersymmetry variation of the bosonic fields involves only Ψ_M, and consequently vanish automatically in the $AdS_2 \times S^2$ background $\Psi_M = 0$. The supersymmetry variation $\delta_\epsilon \Psi_M$ is given by (2.2) and vanishes in the $AdS_2 \times S^2$ background for Killing spinors ϵ satisfying

$$D_M \epsilon = \left[\nabla_M - \frac{1}{2} \gamma \Gamma_M \right] \epsilon = 0 \,. \tag{2.7}$$

The Killing spinors may be written down exactly [17, 18], but we will be content to note that the integrability conditions

$$\begin{aligned} 0 &= [D_\mu, D_\nu]\epsilon = \left[\frac{1}{4} g^{\rho\sigma} R_{\rho\sigma} + \frac{1}{2} \right] \gamma_{\mu\nu}\epsilon \,, \\ 0 &= [D_m, D_n]\epsilon = \left[\frac{1}{4} g^{pq} R_{pq} - \frac{1}{2} \right] \gamma_{mn}\epsilon \end{aligned} \tag{2.8}$$

are consistent with (2.6), implying the existence of the maximal number of solutions for ϵ, namely eight (four complex components).

3. Bosonic Kaluza-Klein spectrum

In this section we calculate the spectrum of bosonic fluctuations around the $AdS_2 \times S^2$ background (2.5) and (2.6). We write the fluctuations as

$$\delta g_{MN} = h_{MN} \,, \qquad \delta A_M = a_M \,, \tag{3.1}$$

and expand the fields in spherical harmonics $Y_{(lm)}$ on S^2, which satisfy

$$\Box_y Y_{(lm)}(y) = -l(l+1) Y_{(lm)}(y) \,, \tag{3.2}$$

($\Box_y \equiv \nabla^m \nabla_m$). The conventionally defined $Y_{(lm)}$ also satisfy the useful identity

$$\nabla_m \nabla_n Y_{(1m)} = -g_{mn} Y_{(1m)} \,. \tag{3.3}$$

We use parentheses to distinguish the SO(3) quantum numbers lm from S^2 indices. The Kaluza-Klein modes will carry SU(2) representation labels (lm) and in particular the bosonic modes will fall into $(2l+1)$-tuplets of SO(3) with charge l. Since the rank of SO(3) is one, all higher tensor harmonics can be expressed in terms of the scalars $Y_{(lm)}$. In particular, the space of vector spherical harmonics is spanned by[2]

$$\nabla_m Y_{(lm)} \,, \qquad \epsilon_{mn} \nabla^n Y_{(lm)} \,, \qquad l \geq 1 \,, \tag{3.4}$$

[2] We have not normalized these basis vectors since we only use their orthogonality. However, the interested reader will find the normalization, and additional details, in [19].

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where we remark for later use that for $l=1$, $K^m_{(m)} = \epsilon^{mn}\nabla_n Y_{(1m)}$ are the **3** of Killing vectors on S^2 and $C^m_{(m)} = \nabla^m Y_{(1m)}$ are the **3** of conformal Killing vectors on S^2. The proof of this assertion relies on equation (3.3). The space of symmetric traceless tensor spherical harmonics is spanned by

$$\nabla_{\{m}\nabla_{n\}}Y_{(lm)}\,, \qquad \nabla_{\{m}\epsilon_{n\}p}\nabla^p Y_{(lm)}\,, \qquad l\geq 2\,. \tag{3.5}$$

Before proceeding, it is necessary to consider carefully the process of fixing the diffeomorphism and U(1) gauge invariance of the action to eliminate unphysical gauge degrees of freedom. We will see that it is possible to fix the "de Donder" gauge

$$\nabla^m h_{\{mn\}} = 0\,, \qquad \nabla^m h_{\mu m} = 0 \tag{3.6}$$

and the "Lorentz" gauge

$$\nabla^m a_m = 0\,, \tag{3.7}$$

but we will find that there are residual gauge transformations preserving (3.6) and (3.7), whose behaviour will be important below.

3.1 Gauge transformations and gauge fixing

The transformations of h_{MN} and a_M under a diffeomorphism $\xi_M(x,y)$ and gauge transformation $\Lambda(x,y)$ are

$$\delta h_{MN} = \nabla_M\xi_N + \nabla_N\xi_M\,, \qquad \delta a_M = \nabla_M\Lambda + \xi^N F_{NM} + \nabla_M(\xi^N A_N)\,. \tag{3.8}$$

We consider first just diffeomorphisms, parametrized by $\xi_M(x,y)$. To fix the de Donder gauge we need ξ_M to satisfy

$$\nabla^m h_{\{mn\}} + (\Box_y + 1)\xi_n = 0\,, \qquad \nabla^m h_{\mu m} + \Box_y\xi_\mu + \nabla_\mu\nabla^m\xi_m = 0\,. \tag{3.9}$$

Thus, we choose

$$\xi_n = -(\Box_y+1)^{-1}\nabla^m h_{\{mn\}}\,, \qquad \xi_\mu = -\Box_y^{-1}(\nabla^m h_{\mu m} + \nabla_\mu\nabla^m\xi_m)\,. \tag{3.10}$$

We remark that there is no problem inverting the required differential operators on S^2 — the expansion of $h_{\{mn\}}$ in terms of symmetric traceless tensor spherical harmonics (3.5) begins at $l=2$ while the only zero modes of $\Box_y + 1$ are the $l=1$ vector spherical harmonics (3.4), and similarly the expansion of $h_{\mu m}$ begins at $l=1$ while the only zero mode of $\Box_y$ is $Y_{(00)}$.

To see if additional gauge-fixing is possible, it is convenient to expand the diffeomorphism parameter $\xi_M(x,y)$ in spherical harmonics:

$$\begin{aligned}\xi_\mu(x,y) &= \xi^{(lm)}_\mu(x)Y_{(lm)}(y)\,,\\ \xi_m(x,y) &= \zeta^{(lm)}(x)\nabla_m Y_{(lm)}(y) + \xi^{(lm)}(x)\epsilon_{mn}\nabla^n Y_{(lm)}(y)\,,\end{aligned} \tag{3.11}$$

where on the right-hand side the repeated (lm) index implies the summation

$$a^{(lm)}Y_{(lm)} \equiv \sum_{l=0}^{\infty}\sum_{m=-l}^{l} a^{(lm)}Y_{(lm)}\,. \tag{3.12}$$

Maintaining the gauge (3.6) requires

$$\nabla^m(\nabla_\mu \xi_m + \nabla_m \xi_\mu) = 0 \tag{3.13}$$

and

$$\nabla^m(\nabla_m \xi_n + \nabla_n \xi_m - g_{mn}\nabla^p \xi_p) = (\Box_y + 1)\xi_n = 0\,. \tag{3.14}$$

The latter implies that only $\xi^{(1m)}(x)$ and $\zeta^{(1m)}(x)$ are non-zero. Plugging this into the former, we find that $\xi^{(00)}_\mu$ is arbitrary while $\xi^{(1m)}_\mu = -\nabla_\mu \zeta^{(1m)}$.

Therefore, we find that the de Donder gauge (3.6) is maintained by residual gauge transformations of the form

$$\begin{aligned}
\xi_\mu(x,y) &= \frac{1}{\sqrt{4\pi}}\xi^{(00)}_\mu(x) - \sum_{m=-1}^{1}\nabla_\mu \zeta^{(1m)}(x)Y_{(1m)}(y)\,,\\
\xi_n(x,y) &= \sum_{m=-1}^{1}\left[\zeta^{(1m)}(x)\nabla_n Y_{(1m)}(y) + \xi^{(1m)}(x)\epsilon_{np}\nabla^p Y_{(1m)}(y)\right].
\end{aligned} \tag{3.15}$$

The residual gauge parameters constitute a **1** and two **3**'s of SO(3). The **1** corresponds to ordinary two-dimensional diffeomorphisms $\xi^{(00)}_\mu$ in AdS_2, and the two **3**'s correspond to the three "Yang-Mills symmetries" $\xi^{(1m)}$ and three "conformal diffeomorphisms" $\zeta^{(1m)}$. We will consider the residual gauge transformations further in section 3.3.

Having explored diffeomorphisms to our satisfaction, we now need to find $\Lambda(x,y)$ so that the Lorentz gauge (3.7) is achieved. This is obviously accomplished via

$$\Lambda(x,y) = -\Box_y^{-1}\nabla^m a_m\,. \tag{3.16}$$

Again there is no problem inverting the laplacean. The residual gauge transformations are those for which $\Box_y \Lambda = 0$, i.e. $\Lambda(x,y) = \Lambda(x)$. These provide the usual two dimensional gauge invariance, as discussed further in section 3.3.

In this subsection we verified that it is possible to fix the de Donder (3.6) and Lorentz (3.7) gauges, and we have found that there are a finite set of residual gauge transformations preserving these gauge choices. These residual gauge transformations differ from the gauge transformations (3.10) and (3.16) used to fix de Donder and Lorentz gauges in an essential way. Finding the latter gauge transformations involved inverting differential operators on S^2, which we showed was always possible to do, whereas fixing the residual gauge degrees of freedom involves inverting differential operators on AdS_2, a process which is not completely well-defined because of ambiguity in the choice of $\Box_x^{-1}$ related to a choice of boundary conditions at the edge of AdS_2. The residual gauge transformations may be used to eliminate the bulk degrees of freedom associated with a given field, but they do not eliminate boundary degrees of freedom associated with that mode. That is, the fields can be gauged away everywhere except on the boundary of AdS_2. The modes killed by imposing the de Donder and Lorentz gauges, on the other hand, have no residual degrees of freedom on the boundary and can be ignored completely.

3.2 Spherical harmonic expansion

We expand the bosonic fields as

$$\begin{aligned}
h_{\mu\nu}(x,y) &= H_{\mu\nu}^{(lm)}(x)Y_{(lm)}(y)\,,\\
h_{\mu m}(x,y) &= B_\mu^{(lm)}(x)\epsilon_{mn}\nabla^n Y_{(lm)}(y) + \tilde{B}_\mu^{(lm)}(x)\nabla_m Y_{(lm)}(y)\,,\\
h_{\{mn\}}(x,y) &= \phi^{(lm)}(x)\nabla_{\{m}\nabla_{n\}}Y_{(lm)}(y) + \tilde{\phi}^{(lm)}(x)\nabla_{\{m}\epsilon_{n\}p}\nabla^p Y_{(lm)}\,,\\
h^m{}_m(x,y) &= \pi^{(lm)}(x)Y_{(lm)}(y)\,,\\
a_\mu(x,y) &= b_\mu^{(lm)}(x)Y_{(lm)}(y)\,,\\
a_m(x,y) &= b^{(lm)}(x)\epsilon_{mn}\nabla^n Y_{(lm)}(y) + \tilde{b}^{(lm)}(x)\nabla_m Y_{(lm)}(y)\,.
\end{aligned} \tag{3.17}$$

Imposing the de Donder (3.6) and Lorentz (3.7) gauges to fix local diffeomorphism and U(1) gauge invariance eliminates $\tilde{B}_\mu$, ϕ, $\tilde{\phi}$, and $\tilde{b}$, leaving

$$\begin{aligned}
h_{\mu\nu}(x,y) &= H_{\mu\nu}^{(lm)}(x)Y_{(lm)}(y)\,,\\
h_{\mu m}(x,y) &= B_\mu^{(lm)}(x)\epsilon_{mn}\nabla^n Y_{(lm)}(y)\,,\\
h_{\{mn\}}(x,y) &= 0\,,\\
h^m{}_m(x,y) &= \pi^{(lm)}(x)Y_{(lm)}(y)\,,\\
a_\mu(x,y) &= b_\mu^{(lm)}(x)Y_{(lm)}(y)\,,\\
a_m(x,y) &= b^{(lm)}(x)\epsilon_{mn}\nabla^n Y_{(lm)}(y)\,.
\end{aligned} \tag{3.18}$$

We note for later use that we can set

$$b^{(00)} = 0\,, \qquad B_\mu^{(00)} = 0 \tag{3.19}$$

since the corresponding vector spherical harmonic vanishes at $l = 0$.

3.3 Residual gauge transformations

Recalling equation (3.15), we now consider separately the action of each of these residual gauge transformations. Under an ordinary diffeomorphism $\xi_\mu^{(00)}$ we have

$$\delta h_{\mu\nu} = \frac{1}{\sqrt{4\pi}}(\nabla_\mu \xi_\nu^{(00)} + \nabla_\nu \xi_\mu^{(00)})\,, \tag{3.20}$$

which is indeed the action of a usual two-dimensional diffeomorphism. In particular, upon comparison with the expansion of $h_{\mu\nu}$ in (3.17), we find

$$\delta H^{(00)}_{\mu\nu} = \nabla_\mu \xi^{(00)}_\nu + \nabla_\nu \xi^{(00)}_\mu \,. \tag{3.21}$$

Thus $\xi^{(00)}_\mu$ provides gauge invariance for $H^{(00)}_{\mu\nu}$, which we will find to be the massless AdS_2 graviton.

The action of $\xi^{(1m)}$ is

$$\delta h_{\mu n} = \sum_{m=-1}^{1} \nabla_\mu \xi^{(1m)} \epsilon_{np} \nabla^p Y_{(1m)} \,, \tag{3.22}$$

which upon comparison with the expansion of $h_{\mu n}$ in (3.17) gives

$$\delta B^{(1m)}_\mu = \nabla_\mu \xi^{(1m)} \,. \tag{3.23}$$

Thus $\xi^{(1m)}$ provides gauge invariance for the SO(3) triplet of vectors $B^{(1m)}_\mu$, which we will find to be massless.

The action of $\zeta^{(1m)}$ on h_{MN} takes the form

$$\begin{aligned} \delta h^n{}_n &= -4 \sum_{m=-1}^{1} \zeta^{(1m)} Y_{(1m)} \,, \\ \delta h_{\mu\nu} &= -2 \sum_{m=-1}^{1} \nabla_\mu \nabla_\nu \zeta^{(1m)} Y_{(1m)} \,, \end{aligned} \tag{3.24}$$

which amounts to

$$\delta \pi^{(1m)} = -4 \zeta^{(1m)} \,, \qquad \delta H^{(1m)}_{\mu\nu} = -2 \nabla_\mu \nabla_\nu \zeta^{(1m)} \,. \tag{3.25}$$

We can use these conformal diffeomorphisms to set some linear combination of $H^{(1m)\mu}{}_\mu$ and $\pi^{(1m)}$ to zero.

Also, the residual gauge transformation $\Lambda(x)$, left undefined by equation (3.10), gives

$$\delta b^{(00)}_\mu = \nabla_\mu \Lambda(x) \tag{3.26}$$

for the vector field $b^{(00)}_\mu$, which we will also find to be massless.

3.4 Equations of motion

The linearized Einstein equations are

$$\begin{aligned} &-\frac{1}{2} \nabla_\mu \nabla_\nu (h^\rho{}_\rho + h^m{}_m) - \frac{1}{2} (\Box_x + \Box_y + 2) h_{\mu\nu} + \nabla_{(\mu} \nabla^\rho h_{\nu)\rho} + g_{\mu\nu} h^\rho{}_\rho = \\ &\qquad\qquad = g_{\mu\nu} (h^m{}_m - \epsilon^{mn} \nabla_m a_n) \,, \\ &-\frac{1}{2} \nabla_\mu \nabla_m \left(h^\nu{}_\nu + \frac{1}{2} h^n{}_n \right) - \frac{1}{2} (\Box_x + \Box_y - 2) h_{\mu m} + \nabla_{(\mu} \nabla^\nu h_{m)\nu} = \\ &\qquad\qquad = \epsilon_m{}^n (\nabla_\mu a_n - \nabla_n a_\mu) \,, \\ &-\frac{1}{2} \nabla_m \nabla_n h^\rho{}_\rho - \frac{1}{2} (\Box_x + \Box_y - 2) h_{mn} + \nabla_{(m} \nabla^\mu h_{n)\mu} = g_{mn} \epsilon^{pq} \nabla_p a_q \,, \end{aligned} \tag{3.27}$$

and the linearized Maxwell equations, written in components, are

$$
\begin{aligned}
0 &= -2\epsilon^{mn}\nabla_m h_{n\mu} + (\Box_x + \Box_y + 1)a_\mu - \nabla_\mu \nabla^\nu a_\nu\,, \\
0 &= 2\epsilon_m{}^n \nabla^\mu h_{\mu n} - \epsilon_{mn}\nabla^n (h^\mu{}_\mu - h^p{}_p) + (\Box_x + \Box_y - 1)a_m - \nabla_m \nabla^\mu a_\mu\,.
\end{aligned} \tag{3.28}
$$

Inserting (3.18) into these equations of motion gives

$$
\begin{aligned}
&\Big[-\frac{1}{2}\nabla_\mu\nabla_\nu(H^{(lm)\rho}{}_\rho + \pi^{(lm)}) - \frac{1}{2}(\Box - (l+2)(l-1))H^{(lm)}_{\mu\nu} + \\
&\qquad + \nabla_{(\mu}\nabla^\rho H^{(lm)}_{\nu)\rho} + g_{\mu\nu}(H^{(lm)\rho}{}_\rho - \pi^{(lm)} + l(l+1)b^{(lm)})\Big] Y_{(lm)} = 0\,,
\end{aligned} \tag{3.29}
$$

$$
\begin{aligned}
&\left[-\frac{1}{2}\nabla_\mu(H^{(lm)\rho}{}_\rho + \frac{1}{2}\pi^{(lm)}) + \frac{1}{2}\nabla^\rho H^{(lm)}_{\rho\mu} + \nabla_\mu b^{(lm)}\right]\nabla_m Y_{(lm)} = \\
&= \left[\frac{1}{2}(\Box - l(l+1) - 1)B^{(lm)}_\mu - \frac{1}{2}\nabla_\mu\nabla^\nu B^{(lm)}_\nu - b^{(lm)}_\mu\right]\epsilon_{mn}\nabla^n Y_{(lm)}\,,
\end{aligned} \tag{3.30}
$$

$$
\begin{aligned}
&\left[-\frac{1}{2}H^{(lm)\rho}{}_\rho\right]\nabla_{\{m}\nabla_{n\}}Y_{(lm)} + \left[\nabla^\mu B^{(lm)}_\mu\right]\nabla_{\{m}\epsilon_{n\}p}\nabla^p Y_{(lm)} + \\
&+ g_{mn}\left[\frac{1}{4}l(l+1)H^{(lm)\rho}{}_\rho - \frac{1}{4}(\Box - l(l+1) - 2)\pi^{(lm)} - l(l+1)b^{(lm)}\right]Y_{(lm)} = 0\,,
\end{aligned} \tag{3.31}
$$

$$
\left[-2l(l+1)B^{(lm)}_\mu + (\Box - l(l+1) + 1)b^{(lm)}_\mu - \nabla_\mu\nabla^\nu b^{(lm)}_\nu\right]Y_{(lm)} = 0\,, \tag{3.32}
$$

$$
\begin{aligned}
&\left[-2\nabla^\mu B^{(lm)}_\mu - \nabla^\mu b^{(lm)}_\mu\right]\nabla_m Y_{(lm)} - \\
&- \left[H^{(lm)\mu}{}_\mu - \pi^{(lm)} - (\Box - l(l+1))b^{(lm)}\right]\epsilon_{mn}\nabla^n Y_{(lm)} = 0\,.
\end{aligned} \tag{3.33}
$$

Each term in brackets must vanish separately (for each (lm)) due to the orthogonality of the various spherical harmonics. However, not all of the equations are in effect for $l = 0$ and $l = 1$. For example, in (3.31), $\nabla_{\{m}\nabla_{n\}}Y_{(lm)}$ vanishes for $l < 2$ so we only get the condition $H^{(lm)\rho}{}_\rho = 0$ for $l \geq 2$. For this reason it is worthwhile to consider the $l = 0$ and $l = 1$ levels separately from $l \geq 2$. First, however, we present a brief but useful diversion regarding massive gravitons.

3.5 A note on massive gravitons in AdS_2

Consider a symmetric traceless tensor $k_{\mu\nu}$ in two dimensions which is also transverse, i.e.

$$
\nabla^\mu k_{\mu\nu} = 0\,. \tag{3.34}
$$

In two dimensions, these assumptions are sufficient to prove that

$$
(\Box - R)k_{\mu\nu} = 0\,, \tag{3.35}
$$

where R is the two-dimensional Ricci scalar, with $R = -2$ for AdS_2. Now suppose that $k_{\mu\nu}$ is to satisfy the equation of motion for a graviton of mass m^2 in AdS [20]

$$(\Box + 2 - m^2)k_{\mu\nu} - 2\nabla_{(\mu}\nabla^\rho k_{\nu)\rho} = 0\,. \tag{3.36}$$

The second term vanishes by virtue of (3.34), and the remaining equation is consistent with (3.35) only if $m^2 = 0$. We conclude that a symmetric traceless tensor field $k_{\mu\nu}$ in AdS_2 which satisfies (3.34) can only satisfy (3.36) if $m^2 = 0$. If $m^2 \neq 0$, the only solution is $k_{\mu\nu} = 0$. We will use this analysis in the following sections to eliminate the massive graviton from the Kaluza-Klein spectrum.

3.6 The $l = 0$ level

Recalling (3.19), we only have to worry about the fields $H^{(00)}_{\mu\nu}$, $\pi^{(00)}$, and $b^{(00)}_\mu$. There are only three equations, coming from the brackets multiplying $Y_{(00)}$ in (3.29), (3.31) and (3.32):

$$\begin{aligned}
0 &= \frac{1}{2}\nabla_\mu\nabla_\nu\left(H^{(00)\rho}{}_\rho + \pi^{(00)}\right) + \frac{1}{2}(\Box + 2)H^{(00)}_{\mu\nu} - \nabla_{(\mu}\nabla^\rho H^{(00)}_{\nu)\rho} - g_{\mu\nu}\left(H^{(00)\rho}{}_\rho - \pi^{(00)}\right), \\
0 &= -\frac{1}{4}(\Box - 2)\pi^{(00)}\,, \\
0 &= (\Box - 1)b^{(00)}_\mu - \nabla_\mu\nabla^\nu b^{(00)}_\nu\,.
\end{aligned} \tag{3.37}$$

The second equation gives a scalar with mass $m^2 = 2$, while the third equation may be written as

$$\nabla^\nu f_{\mu\nu} = 0\,, \qquad f_{\mu\nu} = \nabla_\mu b^{(00)}_\nu - \nabla_\nu b^{(00)}_\mu\,, \tag{3.38}$$

(the shift of the laplacean by -1 comes from the curvature of S^2) so $b^{(00)}_\mu$ is a massless vector, which has the appropriate gauge invariance as a result of (3.26). In two dimensions a massless vector has no propagating degrees of freedom on shell, but there are boundary degrees of freedom which we will tabulate below.

It remains to analyze the third equation in (3.37). There is no constraint on the trace of $H^{(00)}_{\mu\nu}$ so we will consider the trace part and the traceless part separately. First consider turning on a fluctuation with $\pi^{(00)} = 0$ and $H^{(00)}_{\mu\nu} = Hg_{\mu\nu}$. The equation of motion for such a mode is

$$(\Box - 2)H = 0\,, \tag{3.39}$$

which seems to give a scalar field H with mass $m^2 = 2$. However, one can use the residual diffeomorphism invariance (3.21) to gauge H to zero everywhere except on the boundary. Therefore H is not a propagating degree of freedom in the bulk but is present on the boundary.

Now consider the traceless part of $H_{\mu\nu}^{(00)}$. The equation of motion for a fluctuation $H_{\mu\nu}^{(00)}$ with $H^{(00)\rho}{}_{\rho} = 0$ is then

$$(\Box + 2)H_{\mu\nu}^{(00)} - 2\nabla_{(\mu}\nabla^{\rho}H_{\nu)\rho}^{(00)} = -\nabla_{\mu}\nabla_{\nu}\pi^{(00)} - 2g_{\mu\nu}\pi^{(00)} \,. \qquad (3.40)$$

As discussed in the introduction, the scalar field $\pi^{(00)}$ cannot be eliminated from this equation. Nevertheless, the source terms on the right-hand side of (3.40) do not affect the mass of $H_{\mu\nu}^{(00)}$. For $\pi^{(00)} = 0$, (3.40) is precisely the equation of motion for a massless graviton (3.36), and indeed $H_{\mu\nu}^{(00)}$ has the correct AdS_2 diffeomorphism invariance to be the massless graviton as a consequence of (3.21). Because of this gauge invariance, $H_{\mu\nu}^{(00)}$ can be eliminated in the bulk, and the massless graviton has only a boundary degree of freedom.

To summarize: at the $l = 0$ level we find one bulk degree of freedom, a scalar with $m^2 = 2$, and three pure gauge modes: a scalar with $m^2 = 2$, a massless vector, and a massless graviton.

3.7 The $l = 1$ level

At the $l = 1$ level there are seven equations, coming from all of the brackets in (3.29)–(3.33) except for the first two in (3.31). From (3.31) and (3.33) we find

$$\begin{aligned} 0 &= \frac{1}{2}H^{(1m)\rho}{}_{\rho} - \frac{1}{4}(\Box - 4)\pi^{(1m)} - 2b^{(1m)} \,, \\ 0 &= H^{(1m)\rho}{}_{\rho} - \pi^{(1m)} - (\Box - 2)b^{(1m)} \,, \end{aligned} \qquad (3.41)$$

which combine to yield

$$(\Box - 6)(\pi^{(1m)} - 2b^{(1m)}) = 0 \,, \qquad (3.42)$$

so that $\pi^{(1m)} - 2b^{(1m)}$ is a scalar with $m^2 = 6$. Since we can use the residual conformal diffeomorphisms (3.25) to gauge-fix $\pi^{(1m)}$ to zero, there is no other scalar at this level. Since the conformal diffeomorphisms eliminate only bulk degrees of freedom, there is an extra boundary degree of freedom associated to an $l = 1$ "scalar." However, because the field equation for this excitation is missing, we cannot determine its mass (although in section 5 we will use group theory to argue that it must have $m^2 = 0$). The missing equation of motion and the use of the residual conformal diffeomorphisms is discussed by [13] in the context of $AdS_5 \times S^5$, where the boundary degrees of freedom in that case turn out to be the six scalars of the $\mathrm{SU}(2,2|4)$ doubleton.

The equations (3.41) show that $H^{(1m)\rho}{}_{\rho}$ is not an indepedent degree of freedom since it can be algebraically eliminated. Furthermore, one also finds from (3.29) and (3.30)

$$\Box H^{(1m)}_{\mu\nu} - 2\nabla_{(\mu}\nabla^{\rho}H^{(1m)}_{\nu)\rho} = -\nabla_\mu\nabla_\nu\left(H^{\rho(1m)}{}_{\rho} + \pi^{(1m)}\right) + \\ + 2g_{\mu\nu}(H^{\rho(1m)}{}_{\rho} - \pi^{(1m)} + 2b^{(1m)})\,, \tag{3.43}$$

$$0 = -\frac{1}{2}\nabla_\mu\left(H^{(1m)\rho}{}_{\rho} + \frac{1}{2}\pi^{(1m)}\right) + \frac{1}{2}\nabla^\rho H^{(1m)}_{\rho\mu} + \nabla_\mu b^{(1m)}. \tag{3.44}$$

The second of these equations gives two first order constraints on $H^{(1m)}_{\mu\nu}$, enabling one to solve for the remaining two components. Thus $H^{(1m)}_{\mu\nu}$ has no on-shell degrees of freedom in the bulk, as we expect for gravitons in two dimensions. To be more precise, consider a traceless fluctuation of $H^{(1m)}_{\mu\nu}$. The linear differential equations (3.44) will have a specific solution involving $b^{(1m)}$ and $\pi^{(1m)}$, as well as a homogeneous solution $\hat{H}^{(1m)}_{\{\mu\nu\}}$ satisfying

$$\nabla^\rho \hat{H}^{(1m)}_{\{\rho\mu\}} = 0\,. \tag{3.45}$$

But the analysis of section 3.5 (with $m^2 = l(l{+}1) = 2$) shows that $\hat{H}^{(1m)}_{\{\rho\mu\}}$ must vanish, implying that $H^{(1m)}_{\{\mu\nu\}}$ has no independent degrees of freedom but is completely determined in terms of $b^{(1m)}$ and $\pi^{(1m)}$ through the specific solution to (3.44).

Finally, from (3.30), (3.32), and (3.33) we have the three equations

$$\begin{aligned} 0 &= \frac{1}{2}(\Box - 3)B^{(1m)}_\mu - \frac{1}{2}\nabla_\mu\nabla^\nu B^{(1m)}_\nu - b^{(1m)}_\mu\,, \\ 0 &= -4B^{(1m)}_\mu + (\Box - 1)b^{(1m)}_\mu - \nabla_\mu\nabla^\nu b^{(1m)}_\nu\,, \\ 0 &= -2\nabla^\mu B^{(1m)}_\mu - \nabla^\mu b^{(1m)}_\mu\,. \end{aligned} \tag{3.46}$$

We can use the gauge invariance (3.23) of $B^{(1m)}_\mu$ to set $\nabla^\mu B^{(1m)}_\mu = 0$, which we see then sets $\nabla^\mu b^{(1m)}_\mu = 0$ as well. The remaining two equations decouple to give

$$(\Box - 5)(b^{(1m)}_\mu + B^{(1m)}_\mu) = 0\,, \qquad (\Box + 1)(2b^{(1m)}_\mu - B^{(1m)}_\mu) = 0\,, \tag{3.47}$$

which are the equations of motion for vectors of mass $m^2 = 6$ and $m^2 = 0$, respectively. The massless vector has the proper gauge invariance as a result of (3.23).

We have exhausted the content of all seven equations. To summarize, at the $l = 1$ level we find two bulk degrees of freedom, one scalar and one vector each with $m^2 = 6$, and boundary degrees of freedom corresponding to a scalar whose mass we could not determine and one massless vector. Each of these modes is a **3** of SO(3).

3.8 The case $l \geq 2$

For $l \geq 2$ we immediately observe from (3.31) that $H^{(lm)\rho}{}_{\rho} = 0 = \nabla^\mu B^{(lm)}_\mu$, and hence, from (3.33), that $\nabla^\mu b^{(lm)}_\mu = 0$. The two equations involving only the vectors $B^{(lm)}_\mu$ and $b^{(lm)}_\mu$ come from (3.30) and (3.32) and now read

$$(\Box + 1)\begin{pmatrix} B^{(lm)}_\mu \\ b^{(lm)}_\mu \end{pmatrix} = \begin{pmatrix} l(l+1) + 2 & 2 \\ 2l(l+1) & l(l+1) \end{pmatrix}\begin{pmatrix} B^{(lm)}_\mu \\ b^{(lm)}_\mu \end{pmatrix}\,. \tag{3.48}$$

Diagonalizing this mass matrix leads to two infinite towers of massive vector fields with on-shell masses $m^2 = \Box + 1$ (where the shift comes from a curvature term in the Maxwell equations) given by

$$m^2 = \begin{cases} l(l-1) \\ (l+1)(l+2) \end{cases}, \qquad l \geq 2\,. \tag{3.49}$$

The two equations involving only the scalars $\pi^{(lm)}$ and $b^{(lm)}$ coming from (3.31) and (3.33) are

$$\Box \begin{pmatrix} \pi^{(lm)} \\ b^{(lm)} \end{pmatrix} = \begin{pmatrix} l(l+1)+2 & -4l(l+1) \\ -1 & l(l+1) \end{pmatrix} \begin{pmatrix} \pi^{(lm)} \\ b^{(lm)} \end{pmatrix}, \tag{3.50}$$

and hence we have two towers of massive scalars with masses ($m^2 = \Box$)

$$m^2 = \begin{cases} l(l-1) \\ (l+1)(l+2) \end{cases}, \qquad l \geq 2\,. \tag{3.51}$$

We have exhausted the content of all but two equations. From (3.29) and (3.30) we have

$$\begin{aligned} (\Box - (l+2)(l-1))H^{(lm)}_{\mu\nu} - 2\nabla_{(\mu}\nabla^\rho H^{(lm)}_{\nu)\rho} &= -\nabla_\mu\nabla_\nu\pi^{(lm)} + \\ &\quad +2g_{\mu\nu}(-\pi^{(lm)} + l(l+1)b^{(lm)})\,, \\ -\frac{1}{4}\nabla_\mu\pi^{(lm)} + \frac{1}{2}\nabla^\rho H^{(lm)}_{\rho\mu} + \nabla_\mu b^{(lm)} &= 0\,. \end{aligned} \tag{3.52}$$

We now mimick the analysis following (3.43) and (3.44) to conclude that $H^{(lm)}_{\mu\nu}$ has no on-shell degrees of freedom.

3.9 Summary

We separate the bosonic spectrum into the physical (propagating) degrees of freedom in the bulk and those which live only on the boundary. The bulk modes are

scalar	$m^2 = l(l-1)$	$l \geq 2$
scalar	$m^2 = (l+1)(l+2)$	$l \geq 0$
vector	$m^2 = l(l-1)$	$l \geq 2$
vector	$m^2 = (l+1)(l+2)$	$l \geq 1$

(3.53)

where each field represents one on-shell bosonic degree of freedom. The modes which live only on the boundary are

scalar	$m^2 = ?$	$l = 1$
scalar	$m^2 = 2$	$l = 0$
vector	$m^2 = l(l-1) = 0$	$l = 0, 1$
graviton	$m^2 = 0$	$l = 0$

(3.54)

The first scalar has no associated field equation so it is impossible to determine its mass. (In section 5 we will use group theory to argue that the mass of this scalar is $m^2 = 0$.) All of the entries in this table represent pure gauge modes, i.e. the corresponding field can be gauged away everywhere except on the boundary of AdS_2 using the residual gauge invariances disussed in section 3.1.

The tables (3.53) and (3.54) are not very useful. For one thing, the distinction between scalars and vectors makes no sense in two dimensions, where each one has one on-shell degree or freedom. Secondly, the m^2 column doesn't really make sense in AdS_2 — on-shell, m^2 is the quadratic Casimir of the Poincaré group, which is the isometry group of flat space but not of AdS_2. Instead of m^2, we should label our bosonic modes by their quantum numbers (h, q) under the $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SU}(2)$ isometry group of $AdS_2 \times S^2$.

For bosonic fields in AdS_2, the conformal weight h is related to the "mass" (as conventially defined) by the well-known formula [4]

$$h_\pm = \frac{1}{2}(1 \pm \sqrt{1 + 4m^2})\,, \tag{3.55}$$

where one takes the larger root. The SU(2) charge q is simply the l of the associated spherical harmonic $Y_{(lm)}$. Using these rules we find that the $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SU}(2)$ quantum numbers of the bulk modes are

$$(2,0)\,, \qquad 2(k,k)_{k\geq 2}\,, \qquad 2(k+1,k-1)_{k\geq 2}\,, \tag{3.56}$$

and those of the boundary modes are

$$(?,1)\,, \qquad (2,0)\,, \qquad (1,1)\,, \qquad 2(1,0)\,. \tag{3.57}$$

In section 5 we argue that the first mode must be $(1,1)$.

4. Fermionic Kaluza-Klein spectrum

In this section we shall calculate the spectrum of fermionic fluctuations around the $AdS_2 \times S^2$ background $\Psi_M = 0$. The lagrangean (2.1) gives the following equation of motion for Ψ_M in a background with $\Psi_M = 0$,

$$\Gamma^{MNP}\nabla_N \Psi_P + \frac{i}{2}\left(F^{MN} - i\Gamma^5 * F^{MN}\right)\Psi_N = 0\,. \tag{4.1}$$

In the $AdS_2 \times S^2$ background given by (2.5) and (2.6) this equation takes the simple form

$$\Gamma^{MNP} D_N \Psi_P = 0\,, \tag{4.2}$$

with D_N given by (2.7). We decompose the four-dimensional gravitino $\Psi_M(x,y)$ into $\Psi_\mu(x,y) = \psi_\mu(x) \otimes \chi(y)$ and $\Psi_m(x,y) = \psi(x) \otimes \chi_m(y)$, where $\psi_\mu(x)$ and $\psi(x)$ are spacetime gravitini and spinors and $\chi(y)$ and $\chi_m(y)$ are internal (S^2) spinors and vector-spinors. The equation of motion (4.2) couples the spacetime gravitini and spinor degrees of freedom, but they can be decoupled by imposing the condition

$$\Gamma^M \Psi_M = 0 \tag{4.3}$$

to gauge-fix local supersymmetry. Using the equation of motion (4.2), one finds that the condition (4.3) also implies that

$$D_M \Psi^M = \nabla_M \Psi^M = 0\,. \tag{4.4}$$

Using the conditions (4.3) and (4.4), the decoupled equations of motion for ψ_μ and ψ read

$$(\gamma^\nu \nabla_\nu \psi^\mu - \gamma^{\mu\nu}\psi_\nu) \otimes \chi = \psi^\mu \otimes (-\gamma\gamma^m \nabla_m \chi)\,, \tag{4.5a}$$
$$(\gamma^\mu \nabla_\mu \psi) \otimes \chi^m = \psi \otimes (-\gamma\gamma^n \nabla_n \chi^m - \gamma^{mn}\chi_n)\,. \tag{4.5b}$$

4.1 Spinor harmonics on S^2

It will be convenient to expand the fermionic fields in terms of eigenspinors of the 'Dirac' operator $-\gamma\gamma^m\nabla_m$ on S^2. We denote the eigenspinors of this operator by $\eta_{(lm)}(y)$, where l is now a half-integer, $l = 1/2, 3/2, \ldots$, and m ranges as as usual from $-l$ to $+l$ in unit steps. These spinors satisfy

$$-\gamma\gamma^m \nabla_m \eta_{(lm)}(y) = \left(l + \frac{1}{2}\right) \eta_{(lm)}(y)\,, \tag{4.6}$$

and hence

$$\Box_y \eta_{(lm)}(y) = \left[-l(l+1) + \frac{1}{4}\right] \eta_{(lm)}(y)\,. \tag{4.7}$$

Note that $\eta_{(\frac{1}{2},m)}$ are the **2** of Killing spinors on S^2 satisfying

$$D_m \eta_{(\frac{1}{2},m)} = \left[\nabla_m - \frac{1}{2}\gamma\gamma_m\right] \eta_{(1/2,m)} = 0\,, \tag{4.8}$$

and indeed the $\eta_{(lm)}$'s may be constructed from $Y_{(lm)}$ and $\eta_{(\frac{1}{2},m)}$ [13].

4.2 Spinors

The spectrum of spacetime spinors $\psi(x)$ is found by diagonalizing the mass operator

$$M^{(1/2)}_{mn} \equiv -g_{mn}\gamma\gamma^p\nabla_p - \gamma_{mn} \tag{4.9}$$

on S^2. A complete basis of vector-spinors χ_m on S^2 is provided by

$$\gamma_m \eta_{(lm)}\,, \qquad \gamma_m \gamma \eta_{(lm)}\,, \qquad \nabla_m \eta_{(lm)}\,, \qquad \nabla_m \gamma \eta_{(lm)}\,. \tag{4.10}$$

However, this basis includes modes of the form $\chi_m = D_m\chi$ (for some χ), which are spurious since they can be gauged away using the gauge symmetry (2.2). Using the definitions of D_m and $M^{(1/2)}_{mn}$, one can show that

$$D^m M^{(1/2)}_{mn}\chi^n = -\gamma\gamma^n\nabla_n D_m\chi^m \,. \tag{4.11}$$

If one now decomposes χ_m into [21]

$$\chi_m = \varphi_m + D_m\varphi \,, \qquad D^m\varphi_m = 0 \,, \tag{4.12}$$

then (4.11) imples that this decomposition is maintained by the spin-$\frac{1}{2}$ mass operator. Therefore, the eigenspace of this operator naturally decomposes into the space of spurious states with $D_m\chi^m \neq 0$ and the space of physical states which obey

$$D^m\chi_m = 0 \,. \tag{4.13}$$

Viewing D^m as an operator from the four-dimensional space of vector-spinors spanned by (4.10) (at fixed l and m) to the two-dimensional space of spinors spanned by $\eta_{(lm)}$ and $\gamma\eta_{(lm)}$, it is clear that D^m has (at least) a two-dimensional kernel. One can check that the kernel is exactly two-dimensional and that a basis of the physical states satisfying $D^m\chi_m = 0$ is provided by the two vector-spinors

$$\begin{aligned} A^m_{(lm)} &= \gamma\left[\nabla^m - l\gamma\gamma^m\right]\eta_{(lm)} \,, && l \geq \frac{3}{2} \,, \\ B^m_{(lm)} &= \left[\nabla^m + (l+1)\gamma^m\gamma\right]\eta_{(lm)} \,, && l \geq \frac{1}{2} \,, \end{aligned} \tag{4.14}$$

where the restriction on l for A^m arises from noting that $A^m_{(1/2,m)} = \gamma D^m\eta_{(1/2,m)} = 0$. In this basis $M^{(1/2)}_{mn}$ acts diagonally,

$$M^{(1/2)}_{mn}\begin{pmatrix} A^n_{(lm)} \\ B^n_{(lm)} \end{pmatrix} = \begin{pmatrix} (l+\frac{1}{2})A_{m(lm)} \\ -(l+\frac{1}{2})B_{m(lm)} \end{pmatrix} \,. \tag{4.15}$$

Let us summarize $\mathrm{SL}(2,\mathbb{R})\times\mathrm{SU}(2)$ content of these spinors. For each eigenvector-spinor χ_m of $M^{(1/2)}_{mn}$ with eigenvalue λ there will be an AdS_2 spinor ψ satisfying (4.5b)

$$\gamma^\mu\nabla_\mu\psi = \lambda\psi \,. \tag{4.16}$$

The mass of ψ is therefore just $m = \lambda$. The conformal weight h of a spinor field ψ is given by $h = |m| + \frac{1}{2}$. [4] Finally recall that the gravitino Ψ_M is complex, so the spinor ψ is allowed to be complex as well, and therefore represents 2 fermionic degrees of freedom on-shell. We therefore find from (4.15) the following towers of spinors:

$$2\left[(l+1,l)_{l\geq 3/2} \oplus (l+1,l)_{l\geq 1/2}\right] = 2\left(\frac{3}{2},\frac{1}{2}\right) \oplus 4\left(k+\frac{1}{2},k-\frac{1}{2}\right)_{k\geq 2} \,. \tag{4.17}$$

Finally we need to discuss the presence of fermionic boundary degrees of freedom. Although we have not carefully analysed the residual gauge transformations for the fermions, we can identify the affected states by looking for vanishing equations of motion. Vanishing equations of motion signal the presence of states which are subject to a residual gauge symmetry [22]. We saw this for the bosons at the $l = 1$ level with regard to the residual bosonic gauge transformations of section 3.3. The vanishing of $A^m_{(lm)}$ in (4.15) implies through (4.5) that the equation of motion for the associated spacetime spinor ψ vanishes. We identify this ψ as a boundary degree of freedom with $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SU}(2)$ content

$$2\left(\frac{3}{2}, \frac{1}{2}\right), \tag{4.18}$$

(again, ψ counts twice because it is complex). In section (5) we will see that (4.18) falls into an $\mathrm{SU}(1,1|2)$ multiplet with the $l = 1$ scalars whose equation of motion vanishes.

4.3 Gravitini

In two dimensions, gravitini (massless or massive) have no on-shell degrees of freedom. The story is similar to that of the graviton — in the massless case, the gauge symmetry (i.e. supersymmetry) eliminates the states, while in the massive case the equation of motion implies additional constraints which eliminate the states.

The general theory of Kaluza-Klein compactification [23] shows that massless gravitini are in one-to-one correspondence with Killing spinors on the internal manifold, which in this case form a **2** of SU(2).

Taking $\chi = \eta_{(\frac{1}{2},m)}$ in (4.5) gives

$$\gamma^\nu \nabla_\nu \psi^\mu - \gamma^{\mu\nu}\psi_\nu - \lambda\psi^\mu = 0 \tag{4.19}$$

with $\lambda = 1$. The relation between the mass and conformal weight for a spin-$\frac{3}{2}$ field is [4]

$$h = |m| + \frac{1}{2}\,. \tag{4.20}$$

However, the definition of the 'mass' m for a spin-3/2 field to use in this equation has some convention-dependence. A careful analysis [24][3] shows that the field equation (4.19) implies that the m in (4.20) should be $m = \lambda - 1$. Indeed, with this definition of mass, gauge symmetry (i.e. supersymmetry) is restored at $m = 0$. To see this, consider a supersymmetry variation $\delta\psi_\mu = \nabla_\mu\tilde{\eta}$ where $\tilde{\eta}$ is a Killing spinor

[3]Massless Rarita-Schwinger fields were previously considered, in the context of the AdS/CFT correspondence, in Refs. [25, 26].

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on AdS_2. Using $\nabla_\mu \tilde{\eta} = \frac{1}{2}\gamma_\mu \tilde{\eta}$, one easily checks that (4.19) is invariant only when $m = \lambda - 1 = 0$.

The massless gravitino, which has only boundary degrees of freedom, therefore has $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SU}(2)$ content

$$\left(\frac{1}{2}, \frac{1}{2}\right). \tag{4.21}$$

5. $\mathrm{SU}(1,1|2)$ composition of the spectrum

In this section we show how the Kaluza-Klein modes fall into representations of $\mathrm{SU}(1,1|2)$, whose bosonic part is the product of the $\mathrm{SU}(1,1) \cong \mathrm{SL}(2,\mathbb{R})$ isometry of AdS_2 and the $\mathrm{SU}(2)$ isometry of S^2.

The unitary irreducible representations of $\mathrm{SU}(1,1|2)$ are well known [27] (see [6, 4] for a modern review and applications). The short multiplets are labelled by a half-integer $\mathbf{k}$ ($k > 0$) and have the $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SU}(2)$ content [4]

$$\mathbf{k} \equiv (k,k) \oplus 2\left(k + \frac{1}{2}, k - \frac{1}{2}\right) \oplus (k+1, k-1)\,. \tag{5.1}$$

The multiplet includes four states (which are $\mathrm{SL}(2,\mathbb{R})$ primaries), except in the case $k = \frac{1}{2}$ where the last state is missing,

$$\frac{\mathbf{1}}{\mathbf{2}} = \left(\frac{1}{2}, \frac{1}{2}\right) \oplus 2(1,0)\,. \tag{5.2}$$

There are no singleton representations [6].

We now assemble the $\mathrm{SU}(1,1|2)$ multiplets from (3.56), (3.57), (4.18) and (4.17), and (4.21). From the massless graviton, $l = 0$ vector and gravitino we form a single $\frac{\mathbf{1}}{\mathbf{2}}$ of $\mathrm{SU}(1,1|2)$. These fields are just the restriction to the lowest S^2 harmonic of the original fields (g_{MN}, Ψ_M, A_M). They are pure gauge modes in two dimensions. From the infinite towers of scalars, vectors, and spinors we have two towers $2(\mathbf{k})_{k\geq 2}$ of modes with physical degrees of freedom in the bulk of AdS_2. Finally we see that the scalar mode with undetermined h in (3.57) must be $(1,1)$ in order to fit the remaining modes into $\mathrm{SU}(1,1|2)$ representations. Assembling the remaining modes gives $2(\mathbf{1})$. One of these $\mathbf{1}$'s contains fields whose equations of motion were found to vanish, signaling the presence of residual gauge symmetries which eliminate the bulk degrees of freedom of these modes. The other $\mathbf{1}$ contains a bulk scalar, a complex bulk spinor, and a triplet of massless vector fields transforming in the $\mathbf{3}$ of $\mathrm{SU}(2)$. Because we only linearized the equations, we could not see the nonabelian gauge symmetry presumably possessed by these vectors.

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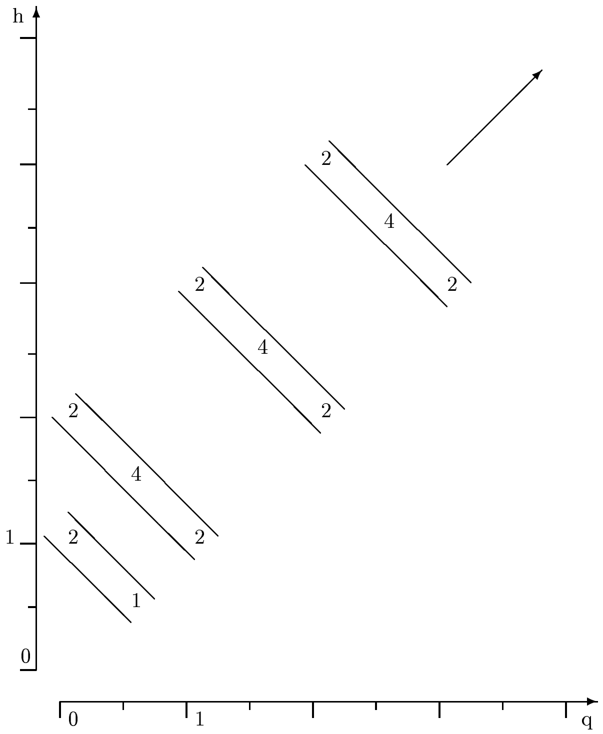

Figure 1: A state at a given value of q forms a $\mathbf{2q+1}$ of SU(2). The top element of the $\mathbf{2q+1}$ has $J_0^3 = +q$ and the bottom element of $J_0^3 = -q$. Those states with $h = J_0^3$ are chiral primaries and those with $h = -J_0^3$ are chiral anti-primaries.

6. Conclusion and discussion

We have determined the Kaluza-Klein spectrum of $\mathcal{N} = 2$, $D = 4$ supergravity on $AdS_2 \times S^2$. The SU$(1,1|2)$ composition of the spectrum is $\mathbf{\frac{1}{2}} \oplus 2(\mathbf{k})_{\mathbb{Z} \ni k \geq 1}$. Furthermore we have categorized the modes according to whether they have on-shell degrees of freedom in the bulk of AdS_2 or whether, because of residual gauge transformations, they may be gauged away everywhere in the bulk of AdS_2 and hence only represent boundary degrees of freedom.

In two dimensions one also encounters modes without any gauge symmetry yet which have no on-shell degrees of freedom in the bulk, such as gravitons and gravitini. We showed explicitly that the equations of motion for the massive gravitons implied that they have no independent degrees of freedom. Our analysis for the gravitini was less careful, but the result must follow by supersymmetry. It is conceivable that one could resurrect these modes (or any other modes one fancies) on the boundary by adding suitable boundary terms to the supergravity action. However they are not required, and indeed our SU$(1,1|2)$ analysis has not found any modes lacking.

Note added. As this work was near completion we learned of similar investigations by J. Lee and S. Lee [29].

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A. Conventions

We use the mostly positive metric signature, $\eta_{MN} = \mathrm{diag}(-1,+1,+1,+1)$. Upper case Latin indices $M, N, \ldots = 0,1,2,3$ run over $AdS_2 \times S^2$, lower case Latin indices $m, n = 2,3$ are internal indices and run over S^2, and Greek indices $\mu, \nu = 0,1$ denote AdS_2 spacetime indices. We use x^μ for coordinates on AdS_2 and y^m for coordinates on S^2. The Levi-Civita symbol ϵ is a true tensor, with $\epsilon_{0123} = \sqrt{-g}$, and $*F^{MN} = \frac{1}{2}\epsilon^{MNPQ}F_{PQ}$. Curly braces around a pair of indices indicate symmetrization with the trace removed, i.e.

$$h_{\{mn\}} = \frac{1}{2}(h_{mn} + h_{nm} - g_{mn}h^p{}_p)\,. \tag{A.1}$$

Let γ^μ and γ^m respectively satisfy the algebras $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ and $\{\gamma^m, \gamma^n\} = 2g^{mn}$. Multiple indices denote antisymmetrization with unit weight, e.g. $\gamma^{mn} = \frac{1}{2}(\gamma^m\gamma^n - \gamma^n\gamma^m)$. The chirality operator on S^2 is given by $\gamma = -\frac{i}{2}\epsilon_{mn}\gamma^{mn}$. A representation of the four-dimensional Dirac algebra $\{\Gamma^M, \Gamma^N\} = 2g^{MN}$ is then furnished by $\Gamma^\mu = \gamma^\mu \otimes \gamma$ and $\Gamma^m = \mathbf{1}_2 \otimes \gamma^m$. In the text we will frequently omit the tensor product symbol and write simply $\Gamma^\mu = \gamma^\mu\gamma$ and $\Gamma^m = \gamma^m$. The four-dimensional chirality operator is $\Gamma^5 = -\frac{i}{4!}\epsilon_{MNPQ}\Gamma^{MNPQ} = \frac{1}{2}\epsilon_{\mu\nu}\gamma^{\mu\nu}\gamma$ (omitting the tensor product symbol).

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